

Wednesday, October 15, Lecture 20: Zeros, poles, residue

Last time... Cauchy's integral formula,
Liouville's thm,

Def: A neighborhood (nbhd) of a point z_0 is a set $U \subseteq \mathbb{C}$ which contains a disc $D_r(z_0) \subseteq U$. nbhd 

Def:

A zero of a holomorphic function f is a point z_0 such that $f(z_0) = 0$.

Ex:

- $f(z) = z^2 + 1 = (z+i)(z-i)$ has two zeros, i and $-i$.
both linear terms. "simple" zeros.

- $g(x) = z^2$ has 1 zero, 0 , of "order" 2.

Lemma 1.40R:

(not identically zero)

Let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic with a zero z_0 .

Then in a nbhd U of z_0 we may write

$$f(z) = (z - z_0)^m g(z) \quad \text{for all } z \in U$$

where

• $g(z) \neq 0$ and holomorphic

• m is unique.

we call m the order of the zero z_0 . Simple = order 1

Note: this is not true for real-differentiable functions.

Ex: (HW not equal to Taylor series)

$f: \mathbb{R} \rightarrow \mathbb{R}$.

$$f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$



There are "too many zeros" here to factor out!
(It just doesn't work).

Meanwhile,

$$f(z) = (z - z_0)^m g(z) \quad g(z) \neq 0 \quad \forall z,$$

means f "looks like" a polynomial

near zeros:

$$f(z) = \begin{array}{ccc} (z - z_0) & (z - z_0)^2 & (z - z_0)^3 \dots \\ \text{graph of } f(z) \text{ near } z_0 \end{array}$$

Pf:
 f is equal to its power series in a disc $D_r(z_0)$, so

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

Let m correspond to first nonzero $a_m \neq 0$.

$$= \sum_{n=m}^{\infty} a_n(z-z_0)^n$$



$m > 0$ since $f(z_0) = 0$. Define

$$g(z) = \sum_{k=0}^{\infty} a_{m+k}(z-z_0)^k$$

then
$$f(z) = (z-z_0)^m g(z)$$

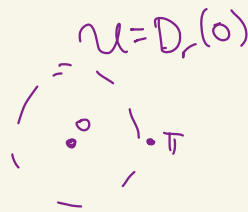
and $g(z_0) = a_m \neq 0$. 

Warning: this is a local description. It may be that $g=0$ outside the nbhd.

Ex: $f(z) = \sin z$ has a simple zero at $z=0$, since

$$\begin{aligned} f(z) &= z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots \\ &= z \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \dots \right) \end{aligned}$$

$\underbrace{\hspace{10em}}_{g(z)}$



nearby $z=0$, e.g. on $U = D_r(0)$ with $0 < r < \pi$, $g(z) \neq 0$. But $g(\pi) = 0$ since $f(\pi) = 0$.

Let's now talk about singularities.



Def: Removable singularity

Let f be holomorphic on a punctured nbhd $U \setminus \{z_0\}$.
We call z_0 a removable singularity if we can define f on Ω so that f is holomorphic.

Ex:

$f(z) = \frac{\sin z}{z}$ is technically undefined at $z_0 = 0$.

But this is easy to fix:

$$f(z) = \begin{cases} \frac{\sin z}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}$$

This is now holomorphic.

Def: Pole

If f is holomorphic on a punctured nbhd $U \setminus \{z_0\}$,
we say it has a pole of order m if

$\frac{1}{f}$ has a zero of order m . If $m=1$, we call it a simple pole.

Ex:

$\frac{1}{z^2}$ has a pole of order 2 at 0, since $\frac{1}{(\frac{1}{z})} = z^2$
has a zero of order 2.

Lemma 1.41 R

A function f has a pole of order m at z_0 iff there is a nbhd $\mathcal{U}$ of z_0 on which

$$f(z) = (z - z_0)^{-m} g(z) \quad \forall z \in \mathcal{U} \setminus \{z_0\}$$

with $g(z) \neq 0$ and holomorphic on $\mathcal{U}$.

Pf:

Use Lemma 1.40 R on $\frac{1}{f}$. 

Thm

If f has a pole of order m at z_0 , then it can be represented uniquely as

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \dots + \frac{a_{-1}}{(z - z_0)} + G(z)$$

where G is holomorphic on a nbhd $\mathcal{U}$ of z_0 and $a_{-m}, a_{-m+1}, \dots, a_{-1} \in \mathbb{C}$ with $a_{-m} \neq 0$.

Pf:

$g(z) = (z - z_0)^m f(z)$ is holomorphic on $\mathcal{U}$ and satisfies

$g(z_0) \neq 0$. Write its power series:

$$\begin{aligned} g(z) &= \sum_{n=0}^{\infty} b_n (z - z_0)^n \\ &= \underbrace{b_0 + b_1(z - z_0) + \dots + b_{m-1}(z - z_0)^{m-1}}_{\neq 0} + \underbrace{\sum_{n=m}^{\infty} b_n (z - z_0)^n}_{= G(z)} \end{aligned}$$

Now, divide both sides by $(z-z_0)^m$:

$$f(z) = \frac{b_0}{(z-z_0)^m} + \frac{b_1}{(z-z_0)^{m-1}} + \dots + \frac{b_{m-1}}{(z-z_0)} + G(z).$$



Def:

Let f have a pole of order m at z_0 , so we may write

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{m+1}}{(z-z_0)^{m-1}} + \dots + \frac{a_{-1}}{(z-z_0)} + G(z)$$

⬇ principal part of f .

↑
holomorphic
near z_0

The residue of f at z_0 is

$$\operatorname{Res}_{z_0} f := a_{-1}$$